COE–3001: Mechanics of deformable bodies

Chapter 5: bending

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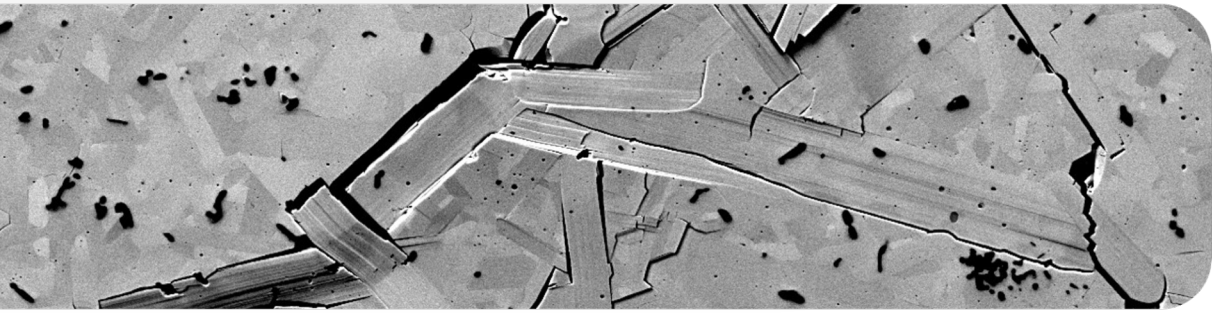

Deformation measurement

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Deformation measure: curvature

❖ **Reminder:**

$$\left\{\mathcal{T}_X^{coh,bending}\right\} = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_{X,z} \end{Bmatrix}$$

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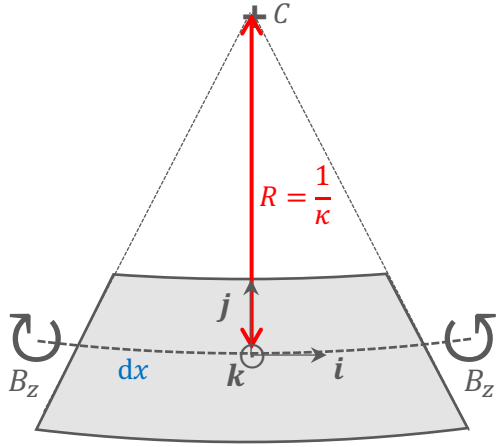

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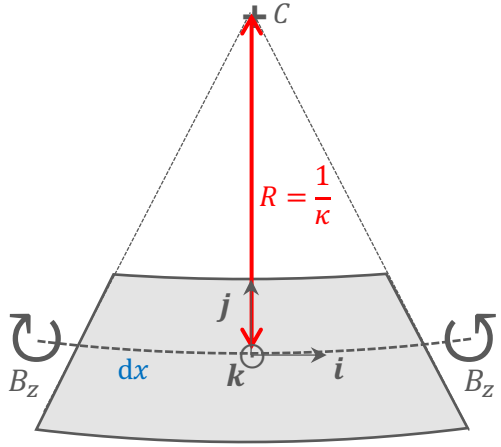
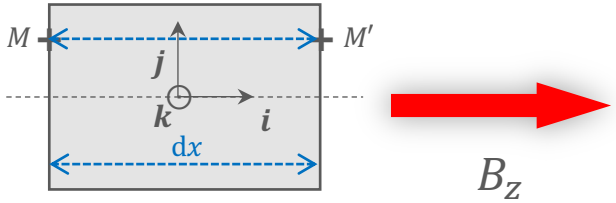
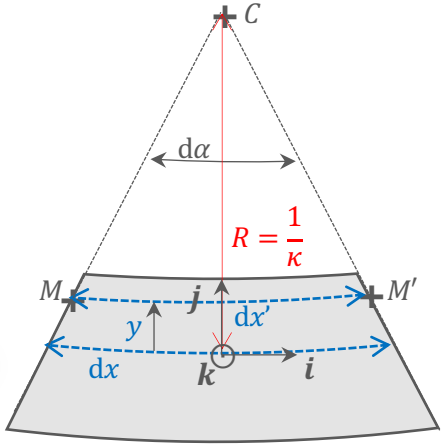


❖ **Infinitesimal rotation of the cross-section:**

$$d\alpha = \frac{dx}{R} \Leftrightarrow \frac{d\alpha}{dx} = \kappa(x)$$
$$dx' = (R - y)d\alpha$$





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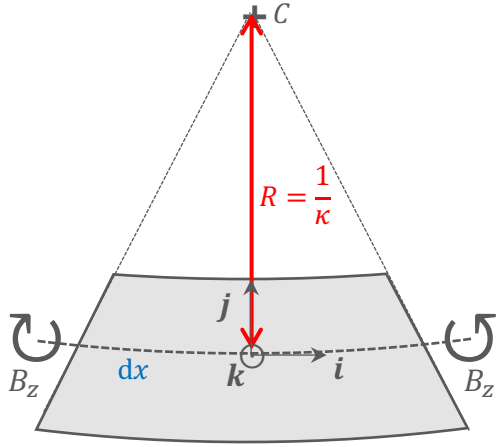

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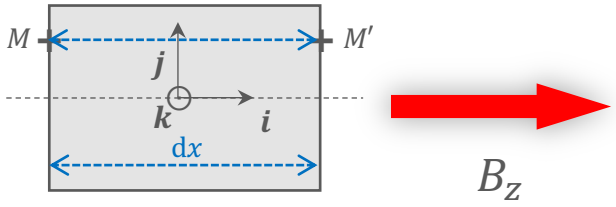

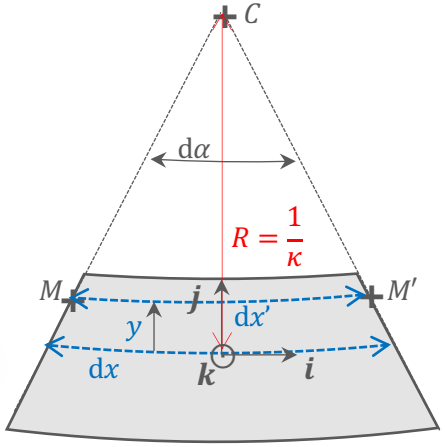

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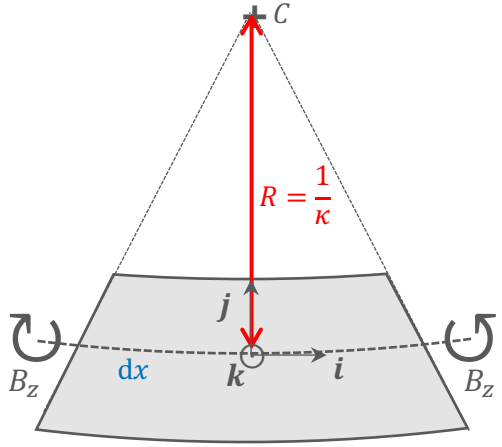

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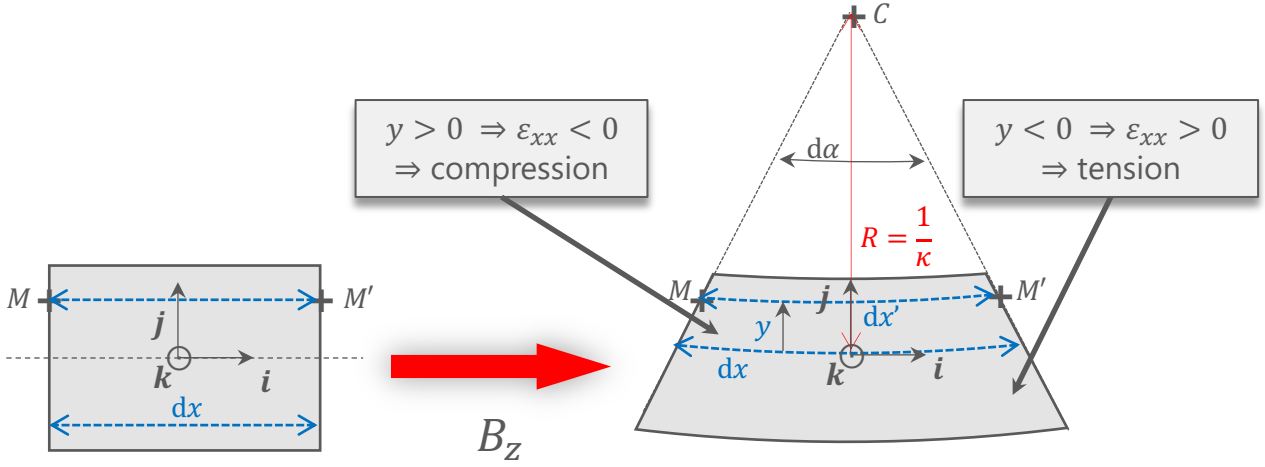
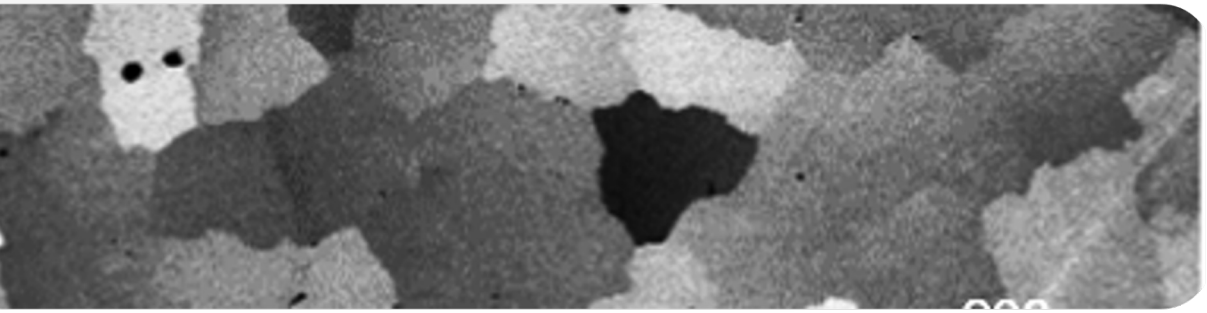




Stress

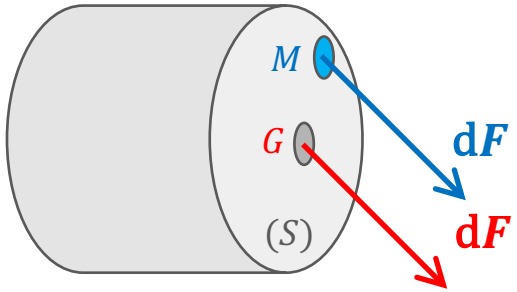
Relationship between ϕ and the internal forces

❖ **Cohesion torsor in M:**

$$\{\mathcal{T}_M^{coh}\} = \begin{Bmatrix} \boldsymbol{R_M} = \mathrm{d}\boldsymbol{F} \\ \boldsymbol{M_M} = \boldsymbol{0} \end{Bmatrix}$$

❖ **Let's transport this torsor into G near M:**

$$\{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R_G} = \boldsymbol{R_M} = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M_G} = \boldsymbol{0} + \boldsymbol{GM}\times\boldsymbol{R_M} \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R_G} = \mathrm{d}\boldsymbol{F} \\ \mathrm{d}\boldsymbol{M_G} = \boldsymbol{GM}\times\mathrm{d}\boldsymbol{F} \end{Bmatrix}$$



$$\text{BUT } \boldsymbol{\phi}(M,\boldsymbol{n}) = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}S} \Rightarrow \mathrm{d}\boldsymbol{F} = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S$$

$$\Rightarrow \{\mathrm{d}\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \mathrm{d}\boldsymbol{R_G} = \boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \\ \mathrm{d}\boldsymbol{M_G} = \boldsymbol{GM}\times\boldsymbol{\phi}(M,\boldsymbol{n})\mathrm{d}S \end{Bmatrix}$$

$$\text{BUT } \boldsymbol{\phi}(M,\boldsymbol{n}=\boldsymbol{i}) = \sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k} \text{ AND } \boldsymbol{GM} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

❖ **After integration over (S):**

$$\{\mathcal{T}_G^{coh}\} = \begin{Bmatrix} \boldsymbol{R_G} = \iint (\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \\ \boldsymbol{M_G} = \iint \boldsymbol{GM}\times(\sigma_{xx}\boldsymbol{i} + \sigma_{xy}\boldsymbol{j} + \sigma_{xz}\boldsymbol{k})\mathrm{d}S \end{Bmatrix} = \begin{Bmatrix} N_G = \iint \sigma_{xx}\mathrm{d}S & T_G = \iint (y\sigma_{xz} - z\sigma_{xy})\mathrm{d}S \\ S_{G,y} = \iint \sigma_{xy}\mathrm{d}S & B_{G,y} = \iint (z\sigma_{xx} - x\sigma_{xz})\mathrm{d}S \\ S_{G,z} = \iint \sigma_{xz}\mathrm{d}S & B_{G,z} = \iint (x\sigma_{xy} - y\sigma_{xx})\mathrm{d}S \end{Bmatrix}$$

↳ Very complex. Additional assumptions are needed about how stresses are distributed over (S).

How to calculate the stress? (normal stress)

❖ **Stress vector in the case of pure bending:**

- No shear stresses on the cross-section

$$\varepsilon_{xx}(x,y) = -y\kappa(x)$$
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❖ **Stress-momentum relationship:**

$$\boldsymbol{M} = \iint \boldsymbol{GM} \times \boldsymbol{\phi}(M, \boldsymbol{i})\boldsymbol{k}\mathrm{d}S = \iint (y\boldsymbol{j} + z\boldsymbol{k}) \times (-Ey\kappa(x)\boldsymbol{i})\mathrm{d}S$$

$$\Rightarrow \boldsymbol{M} = E\kappa(x)\left(\iint y^2\mathrm{d}S\,\boldsymbol{k} - \iint yz\mathrm{d}S\,\boldsymbol{j}\right)$$

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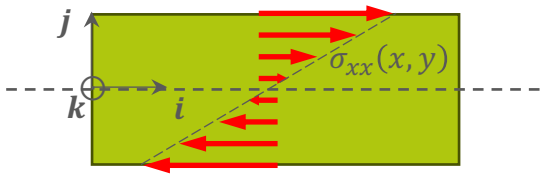
$$\Rightarrow \boldsymbol{M} = E\kappa(x)I_x\boldsymbol{k} \Rightarrow B_{X,z}(x) = E\kappa(x)I_x$$

I_x is the moment of inertia

$$\hookrightarrow \sigma_{xx}(x,y) = -\frac{B_{X,z}(x)}{\iint y^2\mathrm{d}S}y \Rightarrow \sigma_{xx}(x,y)_{max} = -\frac{B_{X,z}(x)}{I_x}y_{max}$$



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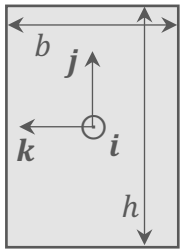

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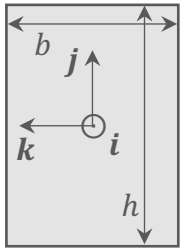

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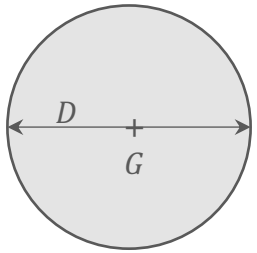

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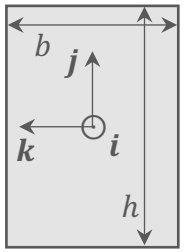

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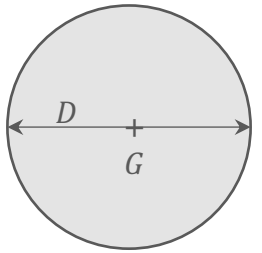
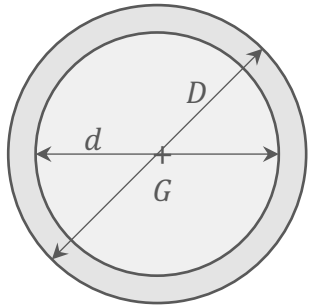


❖ **For a tube:**

$$I_x = \frac{\pi(D^4 - d^4)}{64}$$



↳ I_x depends on the cross-section shape. There are data sheets for complicated shapes.

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❖ **Axial strain:**

- Two adjacent fibers located at different y do not undergo the same axial strain (ε_{xx}):

$$\varepsilon_{xx}(x, y) = -y\kappa(x)$$

⇒ It induces a relative longitudinal sliding between neighboring fibers.

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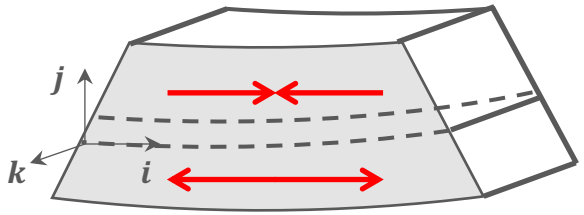

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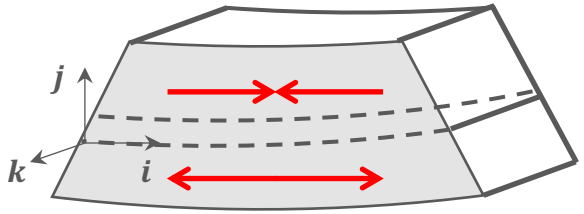

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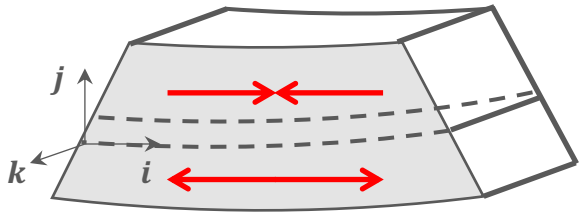

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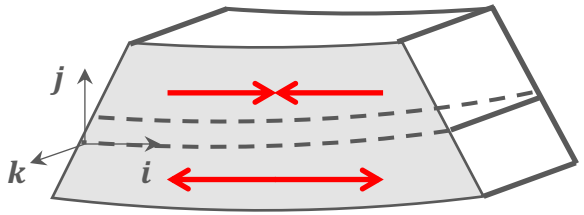

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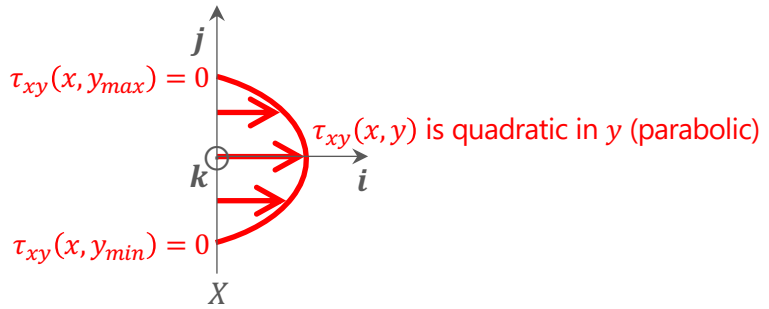
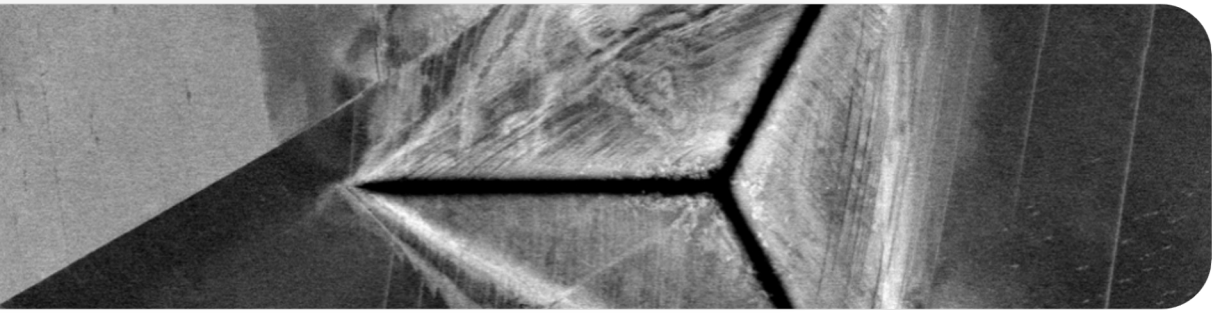
- In the reality, $\tau_{xy}(x,y)$ is non-uniform and must be computed locally.





Beam deflection analysis

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Deflection calculation

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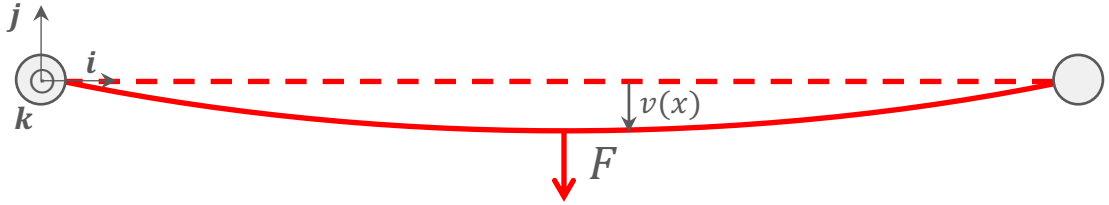

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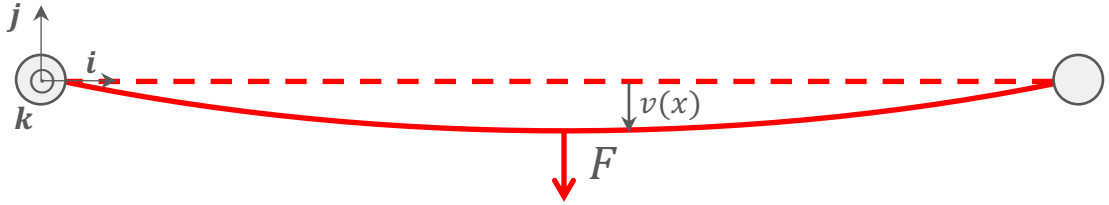

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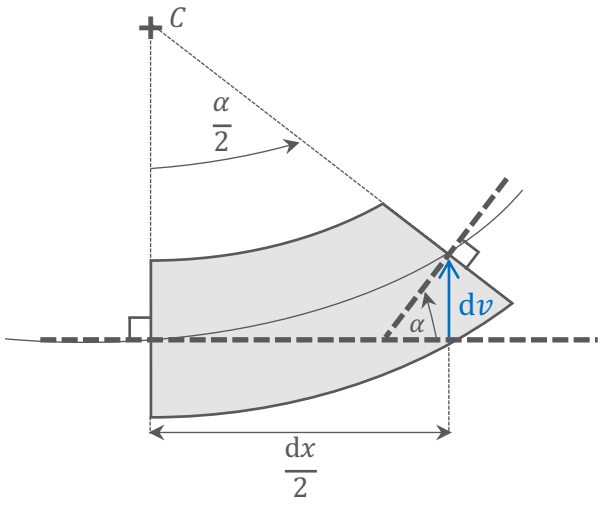

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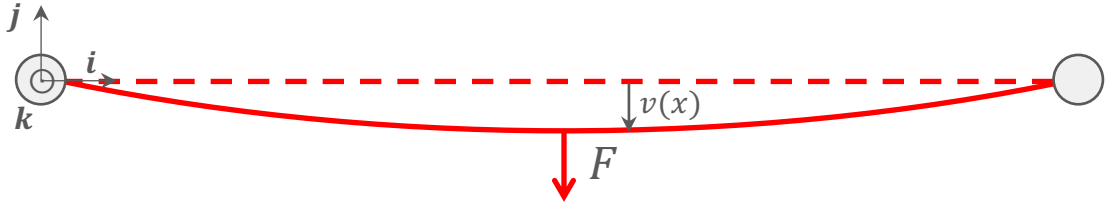

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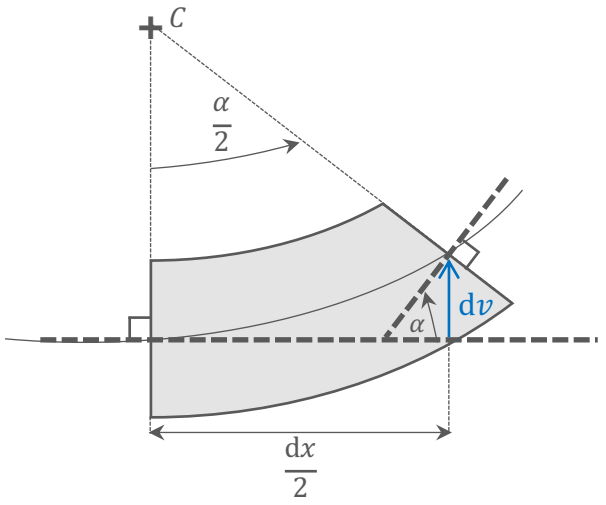

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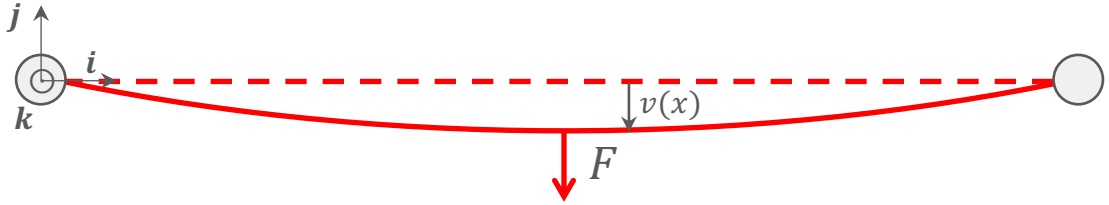

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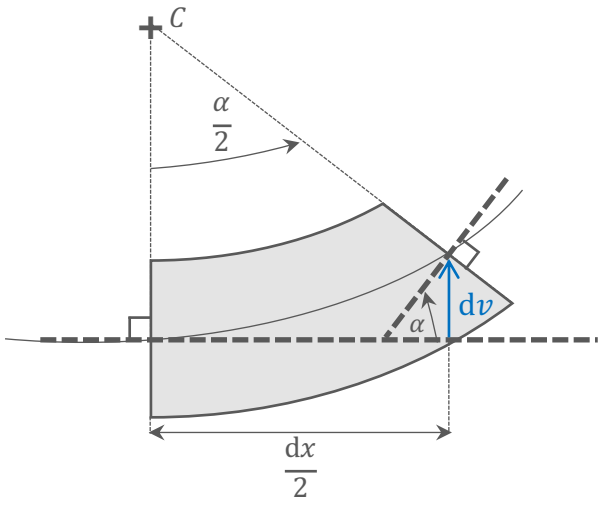

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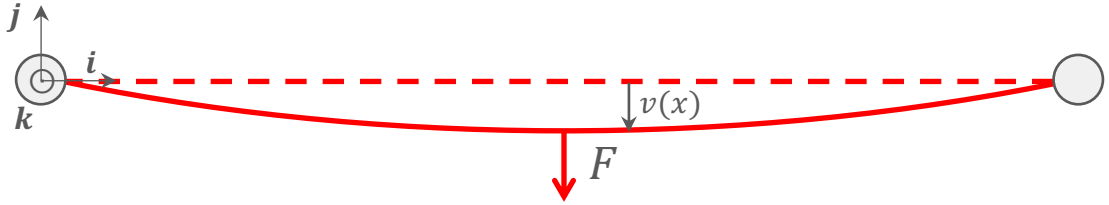

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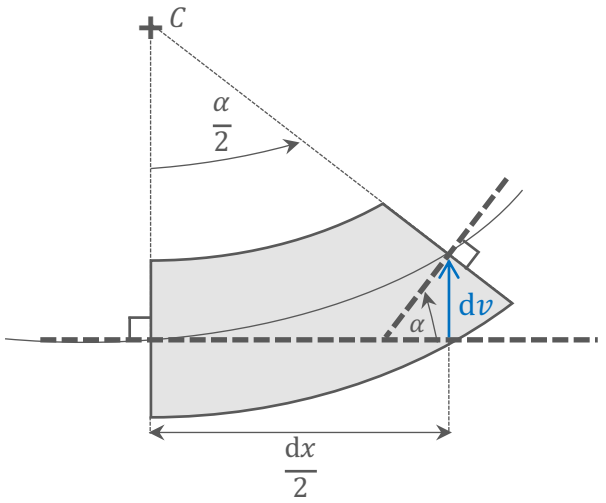
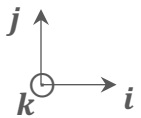


- The deflection $v(x)$ is then computed by successive integrations.

Summary

Loadings	Deformation	Stress	Governing equations	Coefficients	Elasticity limit
Tension	$\varepsilon_{xx} > 0$	$\sigma_{xx} = \frac{N}{A_x} > 0$	$\sigma_{xx} = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e
Compression	$\varepsilon_{xx} < 0$	$\sigma_{xx} = \frac{N}{A_x} < 0$	$\sigma_{xx} = E\varepsilon_{xx}$ $\varepsilon_{zz} = \varepsilon_{yy} = -\nu\varepsilon_{xx}$	E, ν	R_e
Shear	γ_{xy}	$\tau_{xy} = \frac{S_{X,y}}{A_x}$	$\tau_{xy} = G\gamma_{xy}$ $G = \frac{E}{2(1+\nu)}$	G	τ_e
Torsion	θ	$\tau, \sigma = 0$	$\tau = \frac{T_X}{I_G}\rho$ $\tau = G\theta\rho$	G, I_G	τ_e
Bending	κ	σ_{xx}, τ_{xy}	$\sigma_{xx}(x,y) = -\frac{B_z(x)}{I_x}y$ $\tau_{xy}^{avg}(x) = \frac{S_y(x)}{A_{xy}}$ $\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} = \frac{B_z(x)}{EI_x}$	E, I_x	R_e, τ_e



Thanks for your listening!

If you need further information:

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