Mechanics of deformable bodies
COE – 3001
Buckling
Homework #6

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Version: March, 2026

Modeling assumptions

Unless otherwise stated, the following assumptions of strength of materials are adopted throughout this assignment:

- material is continuous, homogeneous, and isotropic,
- deformations are small,
- linear elastic behavior,
- stresses and strains are uniformly distributed,
- loads and supports are perfectly idealized.

Exercise I Questions on the course

Please answer each question to assess your understanding of buckling.

1. Buckling is an instability phenomenon caused by excessive compressive loading.

☐ True ☐ False

2. Buckling can occur before the material reaches its yield strength.

☐ True ☐ False

3. Buckling may be global (as in columns) or local (as in plates or shells).

☐ True ☐ False

4. Geometry often matters more than material strength in buckling.

☐ True ☐ False

5. Euler's theory is valid for short, thick columns where the slenderness ratio is low.

☐ True ☐ False

6. The critical buckling load $F_{c,1}$ decreases when the length L of the column increases.

☐ True ☐ False

7. The lateral deflection $v(x)$ of a buckled beam satisfies the second-order differential equation $\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \omega^2 v(x) = 0$, where $\omega = \sqrt{F/EI_x}$.

☐ True ☐ False

8. The general solution to the buckling differential equation is $v(x) = A\cos(\omega x) + B\sin(\omega x)$.

☐ True ☐ False

9. Applying the boundary condition $v(0) = 0$ to the general solution yields $A = 0$.

☐ True ☐ False

10. For a simply supported (pinned–pinned) beam of length L, the boundary condition $v(L) = 0$ leads to $\sin(\omega L) = 0$, i.e. $\omega L = k\pi$ with $k \in \mathbb{N}^+$.

☐ True ☐ False

11. The buckled shape of a simply supported beam for mode k is $v(x) = B\sin\left(\frac{k\pi}{L}x\right)$.

☐ True ☐ False

12. The critical buckling load is $F_{c,k} = \dfrac{k^2\pi^2 EI_x}{L^2}$, so $F_{c,k}$ is proportional to k^2.

☐ True ☐ False

13. The lowest critical load, which governs structural design, corresponds to $k = 1$.

☐ True ☐ False

14. The critical buckling load is independent of the material's Young's modulus E.

☐ True ☐ False

15. The second moment of area I_x has no effect on the critical buckling load.

☐ True ☐ False

16. Euler's theory requires that the critical stress $\sigma_{cr} = F_{c,1}/A$ remains below the yield strength σ_y.

☐ True ☐ False

17. A column loaded below its critical load $F_{c,1}$ remains straight; lateral deflection appears only when $F \geq F_{c,1}$.

☐ True ☐ False

18. The buckling load of a fixed–fixed column is four times that of an equivalent pinned–pinned column of the same length and cross-section.

☐ True ☐ False

19. Structural collapse due to buckling can occur well before material yielding.

☐ True ☐ False

20. The critical buckling load is inversely proportional to the square of the beam's length: $F_{c,1} \propto 1/L^2$.

☐ True ☐ False

Exercise II Buckling of a steel column in a mezzanine structure

A steel column of circular hollow section is used to support the intermediate floor of an industrial mezzanine. During a structural audit, an engineer must verify that the column does not buckle under the combined weight of the floor and its live load.

The column is made of steel with Young's modulus $E = 210\,\text{GPa}$ and yield strength $\sigma_y = 355\,\text{MPa}$. It has outer diameter $D = 100\,\text{mm}$, wall thickness $t = 5\,\text{mm}$, and length $L = 4.5\,\text{m}$. The column is welded at its base (fixed end) and pinned at its top (free to rotate, no lateral displacement). It is subjected to a compressive force $F = 180\,\text{kN}$.

- The column behaves as a homogeneous, isotropic, linearly elastic beam.
- Small deflections apply and Euler–Bernoulli beam theory is valid.
- The self-weight of the column is neglected.

1. Compute the second moment of area I_x of the hollow circular cross-section about its neutral axis. Recall that for a solid circle of diameter d: $I_{\text{solid}} = \pi d^4/64$.

2. In the course, Euler's theory was derived for a **pinned–pinned** column of length L. For that configuration, both ends are free to rotate, the bending moment is zero at both supports, and there is no horizontal reaction. The governing equation is homogeneous:

$$\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} + \frac{F}{EI_x}\, v(x) = 0.$$

The present column is **fixed–pinned**: the base is clamped (no rotation, no displacement) and the top is pinned (no displacement, free to rotate). These boundary conditions differ from the course in two ways:

- the clamped end develops a non-zero reaction moment,
- equilibrium requires a non-zero horizontal reaction R at the pinned end, which introduces an additional bending moment along the beam.

Starting from the equilibrium of the buckled column and following the same derivation procedure as in the course, determine the effective length L_e for this configuration.

Hint. Let R be the horizontal reaction at the pinned end. Show that the bending moment at section $X(x, v(x))$ is $M(x) = -F\,v(x) + R(L-x)$, write the corresponding differential equation, apply the three boundary conditions $v(0) = 0$, $v'(0) = 0$, $v(L) = 0$, and show that the non-trivial condition reduces to $\tan(\omega L) = \omega L$. The smallest positive root of this equation is $\omega L \approx 4.493$; use this to express L_e and give its numerical value.

3. Starting from the root $\omega L \approx 4.493$ obtained in question 2, derive the expression for the critical load $F_{c,1}$ of the fixed–pinned column. Show that the result can be written in the form $F_{c,1} = \pi^2 EI_x/L_e^2$ and identify L_e. Compute $F_{c,1}$ numerically and express your result in kN.

Note on units. Express E in MPa, I_x in mm^4, and L in mm to obtain $F_{c,1}$ directly in N.

4. Compute the safety factor $s = F_{c,1}/F$. Is the column safe against buckling? What minimum value of s is typically recommended for a structural column?

5. Compute the cross-sectional area A of the hollow section, then the critical stress $\sigma_{cr} = F_{c,1}/A$. Compare σ_{cr} to the yield strength σ_y and conclude on whether the use of Euler's theory is justified. Recall that Euler's theory is valid only if the material remains elastic up to the critical load, i.e. $\sigma_{cr} < \sigma_y$.

6. The course states that *geometry often matters more than material strength.* Illustrate this statement by explaining qualitatively, and quantitatively where possible, how $F_{c,1}$ would change in each of the following cases. For each case, identify which parameter in $F_{c,1} = \pi^2 E I_x / L_e^2$ is affected and by what factor.

   (a) The column length is doubled: $L \to 2L$. How does L_e change, and by what factor is $F_{c,1}$ multiplied?

   (b) The wall thickness is increased from $t = 5\,\text{mm}$ to $t = 8\,\text{mm}$. Compute the new inner diameter d', the new second moment of area I'_x, and the ratio $F_{c,1}^{\text{new}}/F_{c,1}$.

   (c) The boundary conditions are changed to fixed–fixed, for which the effective length is $L_e = L/2$. Starting from the current fixed–pinned configuration ($L_e = 0.7L$), compute the ratio $F_{c,1}^{\text{ff}}/F_{c,1}^{\text{fp}}$ and the new critical load.

Exercise III Buckling of a landing gear strut

During the preliminary design of a light aircraft, an engineer must size the main landing gear strut. The strut is modelled as a slender column subjected to a compressive axial load F upon landing, as illustrated in the course for a simply supported beam loaded at end B.

The strut is made of aluminium alloy with Young's modulus $E = 70\,\text{GPa}$ and yield strength $\sigma_y = 480\,\text{MPa}$. It is a solid circular rod of unknown diameter D and length $L = 600\,\text{mm}$. The maximum landing load transmitted to each strut is $F = 25\,\text{kN}$. A safety factor $s = 2.5$ against buckling is required. The strut is pinned at both ends.

Assume:

- The strut behaves as a homogeneous, isotropic, linearly elastic beam.
- Small deflections apply and Euler–Bernoulli beam theory is valid.
- The self-weight of the strut is neglected.

1. Recall the expression of the critical load $F_{c,1}$ derived in the course for a pinned–pinned column. Identify each term and their physical meaning.

2. The safety factor requirement imposes $F_{c,1} \geq s \cdot F$. Express the minimum required second moment of area $I_{\min}$ as a function of s, F, E, and L.

3. For a solid circular cross-section, $I = \pi D^4/64$. Deduce the minimum diameter $D_{\min}$ satisfying the buckling criterion. Express your result in mm.

4. Compute the critical stress $\sigma_{cr} = F_{c,1}/A$ at $D = D_{\min}$. Compare σ_{cr} to σ_y and conclude on whether Euler's theory is applicable — recall that the theory requires the material to remain in the elastic range up to $F_{c,1}$.

5. The attachment conditions are reconsidered: the top of the strut is now fixed (clamped into the fuselage frame) while the bottom remains pinned, giving $L_e = 0.7\,L$. Compute the new critical load $F_{c,1}$ for the diameter D_{min} found in question **3** and the corresponding new safety factor s. Comment on the effect of the boundary conditions.